

ON WELL-POSEDNESS AND WAVE OPERATOR FOR THE GKDV EQUATION

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ABSTRACT. We consider the generalized Korteweg-de Vries (gKdV) equation $\partial_t u + \partial_x^3 u + \mu \partial_x(u^{k+1}) = 0$, where $k > 4$ is an integer number and $\mu = \pm 1$. We give an alternative proof of the Kenig, Ponce, and Vega result in [9], which asserts local and global well-posedness in $\dot{H}^{s_k}(\mathbb{R})$, with $s_k = (k-4)/2k$. A blow-up alternative in suitable Strichartz-type spaces is also established. The main tool is a new linear estimate. As a consequence, we also construct a wave operator in the critical space $\dot{H}^{s_k}(\mathbb{R})$, extending the results of Côte [2].

1. INTRODUCTION

In this paper, we consider the generalized Korteweg-de Vries (gKdV) equation

$$\begin{cases} \partial_t u + \partial_x^3 u + \mu \partial_x(u^{k+1}) = 0, & x \in \mathbb{R}, t > 0, \\ u(x, 0) = u_0(x), \end{cases} \quad (1.1)$$

where $\mu = \pm 1$ and $k > 4$ is an integer number.

In the particular case $k = 1$, this equation was derived by Korteweg and de Vries [11] in their study of waves on shallow water. Here, we are mainly interested in the case $k > 4$ (the L^2 supercritical case), which is a generalization of the model proposed in [11].

Well-posedness for the Cauchy problem (1.1) is now well understood. We first recall the scaling argument: if u is a solution of (1.1), then, for any $\lambda > 0$, $u_\lambda(x, t) = \lambda^{2/k} u(\lambda x, \lambda^3 t)$ is also a solution with initial data $u_\lambda(x, 0) = \lambda^{2/k} u_0(\lambda x)$. Moreover,

$$\|u_\lambda(\cdot, 0)\|_{\dot{H}^s} = \lambda^{s+2/k-1/2} \|u_0\|_{\dot{H}^s}.$$

Thus, for each k fixed, the scale-invariant Sobolev space is $\dot{H}^{s_k}(\mathbb{R})$, $s_k = 1/2 - 2/k$. As a consequence, the natural Sobolev spaces to study equation (1.1) are $H^s(\mathbb{R})$ with $s \geq s_k$. The well-posedness theory to the Cauchy problem (1.1) was developed by Kenig, Ponce, and Vega [9] (see also Kato [7] for a previous result on this direction). Concerning the small data global theory in the critical Sobolev space $\dot{H}^{s_k}(\mathbb{R})$, it was proved in [9, Theorem 2.15] the following result.

Theorem 1.1. *Let $k > 4$ and $s_k = (k-4)/2k$. Then there exists $\delta_k > 0$ such that for any $u_0 \in \dot{H}^{s_k}(\mathbb{R})$ with*

$$\|D_x^{s_k} u_0\|_2 < \delta_k$$

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there exists a unique solution $u(\cdot)$ of the IVP (1.1) satisfying

$$u \in C(\mathbb{R} : \dot{H}^{s_k}(\mathbb{R})), \quad (1.2)$$

$$\|D_x^{s_k} u\|_{L_x^5 L_t^{10}} < \infty, \quad (1.3)$$

$$\|D_x^{s_k} u_x\|_{L_x^\infty L_t^2} < \infty, \quad (1.4)$$

and

$$\|D_x^{\alpha_k} D_t^{\beta_k} u\|_{L_x^{p_k} L_t^{q_k}} < \infty, \quad (1.5)$$

where

$$\alpha_k = \frac{1}{10} - \frac{2}{5k}, \quad \beta_k = \frac{3}{10} - \frac{6}{5k}, \quad (1.6)$$

$$\frac{1}{p_k} = \frac{2}{5k} + \frac{1}{10}, \quad \frac{1}{q_k} = \frac{3}{10} - \frac{4}{5k}. \quad (1.7)$$

Furthermore, the map $u_0 \mapsto u(t)$ from $\{u_0 \in \dot{H}^{s_k}(\mathbb{R}) : \|D_x^{s_k} u_0\|_2 < \delta_k\}$ into the class defined by (1.2)-(1.5) is Lipschitz.

The method to prove Theorem 1.1 combines smoothing effects and Strichartz-type estimates together with the Banach contraction principle. This result shows to be sharp in view of the work due to Birnir, Kenig, Ponce, Svanstedt, and Vega [1].

One of the main goals of this paper is to reprove the above theorem without using any norm that involves derivative in the time variable. To this end, we introduce a new linear estimate (see Lemma 2.5 below) which allow us to obtain the following result.

Theorem 1.2. *Let $k \geq 4$ and $s_k = (k - 4)/2k$. Given $u_0 \in \dot{H}^{s_k}(\mathbb{R})$, assume*

$$\|D_x^{s_k} u_0\|_{L_x^2} \leq K < \infty. \quad (1.8)$$

There exists $\delta = \delta(K) > 0$ such that if

$$\|U(t)u_0\|_{L_x^{5k/4} L_t^{5k/2}} < \delta, \quad (1.9)$$

then there exists a unique solution u of the integral equation

$$u(t) = U(t)u_0 - \mu \int_0^t U(t-t') \partial_x (u^{k+1})(t') dt' \quad (1.10)$$

such that

$$\|u\|_{L_x^{5k/4} L_t^{5k/2}} \leq 2\delta \quad \text{and} \quad \|D_x^{s_k} u\|_{L_x^5 L_t^{10}} < 2cK. \quad (1.11)$$

The solution also satisfies,

$$\|D_x^{s_k} u\|_{L_t^\infty L_x^2} < 2cK. \quad (1.12)$$

Moreover, there exist $f_\pm \in \dot{H}^{s_k}(\mathbb{R})$ such that

$$\lim_{t \rightarrow \pm\infty} \|D_x^{s_k} (u(t) - U(t)f_\pm)\|_{L_x^2} = 0. \quad (1.13)$$

It is worth to mention that the question of how small the initial data should be to imply global well-posedness in the energy space $H^1(\mathbb{R})$ have been recently addressed by Farah, Linares, and Pastor [4] where sufficient conditions have been obtained.

Without imposing any smallness restriction on the initial data, a local version of Theorem 1.1 is also available in [9, Theorem 2.17]. Here, following the same strategy of Theorem 1.2, we are able to prove the following local well-posedness result

Theorem 1.3. *Let $k \geq 4$ and $s_k = (k - 4)/2k$. Given $u_0 \in \dot{H}^{s_k}(\mathbb{R})$ there exist $T = T(u_0)$ and a unique solution u of the integral equation (1.10) satisfying*

$$\|D_x^{s_k} u\|_{L_{[0,T]}^\infty L_x^2} + \|u\|_{L_x^{5k/4} L_{[0,T]}^{5k/2}} + \|D_x^{s_k} u\|_{L_x^5 L_{[0,T]}^{10}} < \infty. \quad (1.14)$$

Furthermore, given $T' \in (0, T)$ there exists a neighborhood V of u_0 in $\dot{H}^{s_k}(\mathbb{R})$ such that the map $u_0 \mapsto \tilde{u}(t)$ from V into the class defined by (1.14) in the time interval $[0, T']$ is Lipschitz.

Note that the previous result asserts that the existence time depends on the initial data itself and not only on its norm. Our next theorem is concerned with the behavior of the local solution near the possible blow-up time. This is inspired by the results in [10, Theorem 1.2]. Let $T^* = T^*(u_0)$ be the maximum time of existence for the unique solution u of the integral equation (1.10) with initial data $u_0 \in \dot{H}^{s_k}(\mathbb{R})$. When $T^* < \infty$, we prove that always $\|u\|_{L_x^{5k/4} L_{[0,T^*]}^{5k/2}} = \infty$. On the other hand, a direct application of Theorem 1.3 implies that either the $\dot{H}^{s_k}(\mathbb{R})$ -norm of $u(t)$ blows-up in time or the $\dot{H}^{s_k}(\mathbb{R})$ - $\lim_{t \uparrow T^*} u(t)$ does not exist. However, even in the case when the $\dot{H}^{s_k}(\mathbb{R})$ -norm of the solution $u(t)$ does not blow-up (this is the case when $k = 4$ by the mass conservation law) we established blow-up for the Strichartz norm that appears in (2.28). Our result reads as follows.

Theorem 1.4. *Assume $k \geq 4$ and $s_k = (k - 4)/2k$. Suppose $u_0 \in \dot{H}^{s_k}(\mathbb{R})$ and $T^* = T^*(u_0)$ be the maximum time of existence given in Theorem 1.3. If $T^* < \infty$ then*

$$\|u\|_{L_x^{5k/4} L_{[0,T^*]}^{5k/2}} = \infty. \quad (1.15)$$

Moreover, if $\sup_{t \in [0, T^)} \|D_x^{s_k} u(t)\|_{L_x^2} = K < \infty$, we also have*

$$\|D_x^{2/3k} u\|_{L_x^{3k/2} L_{[0,T^*]}^{3k/2}} = \infty. \quad (1.16)$$

Remark 1.5. *In the limit case $k = 4$ we recover the result in [10, Theorem 1.2].*

Next we turn to the construction of the wave operator associated to the equation (1.1). This is the reciprocal problem of the scattering theory, which consists in constructing a solution with a prescribed scattering state. Roughly speaking, for a given profile V (regardless of its size), one looks for a solution of the nonlinear problem $u(t)$, defined for large enough t , such that

$$\lim_{t \rightarrow \infty} \|u(t) - u_V(t)\|_Y = 0,$$

where $u_V(t)$ is the solution of the linear problem with initial data V , and Y stands for a suitable Banach space. Solving the latter problem is also known as the construction of a wave operator.

This question was studied in Besov Spaces for the generalized Boussinesq equation in [3] and in Sobolev or weighted Sobolev spaces for Schrödinger equations by Ginibre, Ozawa, and Velo [5, 6] and for generalized Korteweg-de Vries equations by Côte [2]. In this last paper, the author introduced two different approaches to deal with this problem. The case $k > 4$ was treated in the weighted Sobolev space setting. On the other hand, the case $k = 4$ (L^2 -critical KdV equation) is simpler since the fixed point problem is in fact very similar to that of the Cauchy problem treated by Kenig, Ponce and Vega [9] in their small data global existence theorem (see Theorem 1.1). In this paper, using the new linear estimate (Lemma 2.5 below), we show that we can also apply the same approach to the case $k > 4$, extending the Côte's result to the classical Sobolev spaces. Our theorem is the following

Theorem 1.6. *Let $k \geq 4$ and $s_k = (k - 4)/2k$. For any $v \in \dot{H}^{s_k}(\mathbb{R})$, let $u_v(t)$ be the solution of the linear problem associated with (1.1) with initial data v . Then, there exist $T_0 = T_0(v)$ and $u \in C([T_0, \infty) : \dot{H}^{s_k}(\mathbb{R}))$ solution of the IVP (1.1) satisfying*

$$\lim_{t \rightarrow \infty} \|u(t) - u_v(t)\|_{\dot{H}^{s_k}(\mathbb{R})} = 0. \quad (1.17)$$

Moreover, $u(\cdot)$ is unique in the class

$$\left\{ u \in L_t^\infty \dot{H}^{s_k}(\mathbb{R}) / D_x^{s_k} u \in L_x^5 L_T^{10}, u \in L_x^{5k/4} L_T^{5k/2} \right\}.$$

The plan of this paper is as follows. In the next section we introduce some notation and prove the linear estimates related to our problem. In Section 3 we prove Theorems 1.2 and 1.4. Next, in Section 4, we show Theorem 1.6.

2. NOTATION AND PRELIMINARIES

Let us start this section by introducing the notation used throughout the paper. We use c to denote various constants that may vary line by line. Given any positive numbers a and b , the notation $a \lesssim b$ means that there exists a positive constant c such that $a \leq cb$.

We use $\|\cdot\|_{L^p}$ to denote the $L^p(\mathbb{R})$ norm. If necessary, we use subscript to inform which variable we are concerned with. The mixed norms $L_t^q L_x^r$, $L_{[a,b]}^q L_x^r$ and $L_T^q L_x^r$ of $f = f(x, t)$ are defined, respectively, as

$$\|f\|_{L_t^q L_x^r} = \left(\int_{-\infty}^{+\infty} \|f(\cdot, t)\|_{L_x^r}^q dt \right)^{1/q}, \quad \|f\|_{L_t^q L_x^r} = \left(\int_a^b \|f(\cdot, t)\|_{L_x^r}^q dt \right)^{1/q}$$

and

$$\|f\|_{L_T^q L_x^r} = \left(\int_T^{+\infty} \|f(\cdot, t)\|_{L_x^r}^q dt \right)^{1/q}$$

with the usual modifications when $q = \infty$ or $r = \infty$. Similarly, we also define the norms in the spaces $L_x^r L_t^q$, $L_x^r L_{[a,b]}^q$ and $L_x^r L_T^q$.

The spatial Fourier transform of $f(x)$ is given by

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x) dx.$$

The class of Schwartz functions is represented by $\mathcal{S}(\mathbb{R})$. We shall also define D_x^s and J_x^s to be, respectively, the Fourier multiplier with symbol $|\xi|^s$ and $\langle \xi \rangle^s = (1 + |\xi|)^s$. In this case, the norm in the Sobolev spaces $H^s(\mathbb{R})$ and $\dot{H}^s(\mathbb{R})$ are given, respectively, by

$$\|f\|_{H^s} \equiv \|J_x^s f\|_{L_x^2} = \|\langle \xi \rangle^s \hat{f}\|_{L_\xi^2}, \quad \|f\|_{\dot{H}^s} \equiv \|D_x^s f\|_{L_x^2} = \| |\xi|^s \hat{f} \|_{L_\xi^2}.$$

Let us present now some useful lemmas and inequalities. In what follows, $U(t)$ denotes the linear propagator associated with the gKdV equation, that is, for any function u_0 , $U(t)u_0$ is the solution of the linear problem

$$\begin{cases} \partial_t u + \partial_x^3 u = 0, & x \in \mathbb{R}, t \in \mathbb{R}, \\ u(x, 0) = u_0(x). \end{cases} \quad (2.18)$$

We begin by recalling the results necessary to prove Theorem 1.2.

Lemma 2.1. *Let p, q , and α be such that*

$$-\alpha + \frac{1}{p} + \frac{3}{q} = \frac{1}{2}, \quad -\frac{1}{2} \leq \alpha \leq \frac{1}{q}. \quad (2.19)$$

Then

$$\|D_x^\alpha U(t)u_0\|_{L_t^q L_x^p} \lesssim \|u_0\|_{L_x^2}. \quad (2.20)$$

Also, if

$$\frac{1}{p} + \frac{1}{2q} = \frac{1}{4}, \quad \alpha = \frac{2}{q} - \frac{1}{p}, \quad 1 \leq p, q \leq \infty, \quad -\frac{1}{4} \leq \alpha \leq 1. \quad (2.21)$$

Then,

$$\|D_x^\alpha U(t)u_0\|_{L_x^p L_t^q} \lesssim \|u_0\|_{L_x^2}. \quad (2.22)$$

Proof. See [8, Theorem 2.1] and [10, Proposition 2.1]. \square

Remark 2.2. Note that when $(p, q) = (\infty, 2)$ in (2.22) we have the sharp version of the Kato smoothing effect

$$\|\partial_x U(t)u_0\|_{L_x^\infty L_t^2} \lesssim \|u_0\|_{L_x^2}. \quad (2.23)$$

Moreover, the dual version of (2.23) reads as follows:

$$\|\partial_x \int_0^t U(t-t')g(\cdot, t')dt'\|_{L_x^2} \lesssim \|g\|_{L_x^1 L_t^2}. \quad (2.24)$$

(see [9, Theorem 3.5]).

Lemma 2.3. Assume

$$\frac{1}{4} \leq \alpha < \frac{1}{2}, \quad \frac{1}{p} = \frac{1}{2} - \alpha. \quad (2.25)$$

Then,

$$\|D_x^{-\alpha} U(t)u_0\|_{L_x^p L_t^\infty} \lesssim \|u_0\|_{L_x^2}. \quad (2.26)$$

Proof. See [9, Lemma 3.29]. \square

We can also obtain the following particular cases of the Strichartz estimates in the critical Sobolev space $\dot{H}^{s_k}(\mathbb{R})$.

Corollary 2.4. Let $k \geq 4$ and $s_k = (k-4)/2k$. Then

$$\|D_x^{-1/k} U(t)u_0\|_{L_x^k L_t^\infty} \lesssim \|D^{s_k} u_0\|_{L_x^2}. \quad (2.27)$$

and

$$\|D_x^{2/3k} U(t)u_0\|_{L_x^{3k/2} L_t^{3k/2}} \lesssim \|D^{s_k} u_0\|_{L_x^2}. \quad (2.28)$$

Proof. The estimate (2.27) is a particular case of (2.26). On the other hand, by Sobolev Embedding and inequality (2.20) with $q = 3k/2$ and $\alpha = 2/3k$, we obtain

$$\|D_x^{2/3k} U(t)u_0\|_{L_x^{3k/2} L_t^{3k/2}} \lesssim \|D_x^{2/3k} U(t)D^{s_k} u_0\|_{L_t^{3k/2} L_x^p} \lesssim \|D^{s_k} u_0\|_{L_x^2},$$

where $\frac{1}{p} = \frac{1}{2} + \frac{2}{3k} - \frac{2}{k}$, which implies (2.28). \square

Our next lemma is the fundamental tool to prove Theorem 1.2.

Lemma 2.5. Let $k \geq 4$ and $s_k = (k-4)/2k$. Then

$$\|U(t)u_0\|_{L_x^{5k/4} L_t^{5k/2}} \lesssim \|D^{s_k} u_0\|_{L_x^2}. \quad (2.29)$$

Proof. Interpolate the inequalities (2.28) and (2.27). \square

Next, we recall the following integral estimates.

Lemma 2.6. If $p_i, q_i, \alpha_i, i = 1, 2$, satisfy (2.21), then

$$\|D_x^{\alpha_1} \int_0^t U(t-t')g(\cdot, t')dt'\|_{L_x^{p_1} L_t^{q_1}} \lesssim \|D_x^{-\alpha_2} g\|_{L_x^{p_2'} L_t^{q_2'}},$$

where p'_2 and q'_2 are the Hölder conjugate of p_2 and q_2 , respectively. Moreover, assume (p_1, q_1, α_1) satisfies (2.21) and (p_2, α_2) satisfies (2.25). Then,

$$\|D^{-\alpha_2} \int_0^t U(t-t')g(\cdot, t')dt'\|_{L_x^{p_2} L_t^\infty} \lesssim \|D^{-\alpha_1} g\|_{L_x^{p'_1} L_t^{q'_1}}. \quad (2.30)$$

Proof. Use the duality and TT^* arguments combined with Lemmas 2.1 and 2.3 (see also [10, Proposition 2.2]). \square

Applying the same ideas as the previous lemma together with Lemma 2.5 and (2.23) we also have.

Corollary 2.7. *Let $k \geq 4$ and $s_k = (k-4)/2k$, then the following estimate holds:*

$$\|\partial_x \int_0^t U(t-t')g(\cdot, t')dt'\|_{L_x^{5k/4} L_t^{5k/2}} \lesssim \|D_x^{s_k} g\|_{L_x^1 L_t^2}.$$

Remark 2.8. *In all of the above inequalities we can replace the integral \int_0^t by \int_t^∞ . This kind of estimate will be used in Section 4.*

Finally, we have the following estimates for fractional derivatives.

Lemma 2.9. *Let $0 < \alpha < 1$ and $p, p_1, p_2, q, q_1, q_2 \in (1, \infty)$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$. Then,*

(i)

$$\|D_x^\alpha(fg) - fD_x^\alpha g - gD_x^\alpha f\|_{L_x^p L_t^q} \lesssim \|D_x^\alpha f\|_{L_x^{p_1} L_t^{q_1}} \|g\|_{L_x^{p_2} L_t^{q_2}}.$$

The same still holds if $p = 1$ and $q = 2$.

(ii)

$$\|D_x^\alpha F(f)\|_{L_x^p L_t^q} \lesssim \|D_x^\alpha f\|_{L_x^{p_1} L_t^{q_1}} \|F'(f)\|_{L_x^{p_2} L_t^{q_2}}.$$

Proof. See [9, Theorems A.6, A.8, and A.13]. \square

3. GLOBAL WELL-POSEDNESS AND THE BLOW-UP ALTERNATIVE

Our aim in this section is to establish Theorems 1.2 and 1.4.

Proof of Theorem 1.2. As usual, our proof is based on the contraction mapping principle. Hence, we define

$$X_{a,b}^k = \{u \in C(\mathbb{R}; \dot{H}^{s_k}(\mathbb{R})) : \|u\|_{L_x^{5k/4} L_t^{5k/2}} \leq a \quad \text{and} \quad \|D_x^{s_k} u\|_{L_t^\infty L_x^2} + \|D_x^{s_k} u\|_{L_x^5 L_t^{10}} \leq b\}.$$

On $X_{a,b}^k$ consider the integral operator

$$\Phi(u)(t) := U(t)u_0 - \mu \int_0^t U(t-t')\partial_x(u^{k+1})(t')dt'. \quad (3.31)$$

We need to show that

$$\Phi : X_{a,b}^k \rightarrow X_{a,b}^k$$

is a contraction, for an appropriated choice of the parameters $a > 0$ and $b > 0$.

We first estimate the $\|\cdot\|_{L_x^{5k/4} L_t^{5k/2}}$ -norm. From Corollary 2.7,

$$\|\Phi(u)\|_{L_x^{5k/4} L_t^{5k/2}} \leq \|U(t)u_0\|_{L_x^{5k/4} L_t^{5k/2}} + c\|D_x^{s_k}(u^{k+1})\|_{L_x^1 L_t^2}.$$

The estimate of the term $\|D_x^{s_k}(u^{k+1})\|_{L_x^1 L_t^2}$ can be found in [9] (see equation (6.1)). For the sake of completeness we will also perform it here. Indeed, applying the fractional derivative rule in Lemma 2.9-(i), we obtain

$$\begin{aligned} \|D_x^{s_k}(u^{k+1})\|_{L_x^1 L_t^2} &\lesssim \|u^k\|_{L_x^{5/4} L_t^{5/2}} \|D_x^{s_k} u\|_{L_x^5 L_t^{10}} + \|u D_x^{s_k}(u^k)\|_{L_x^1 L_t^2} \\ &\lesssim \|u\|_{L_x^{5k/4} L_t^{5k/2}}^k \|D_x^{s_k} u\|_{L_x^5 L_t^{10}} + \|u\|_{L_x^{5k/4} L_t^{5k/2}} \|D_x^{s_k}(u^k)\|_{L_x^{p_0} L_t^{q_0}} \\ &\lesssim \|u\|_{L_x^{5k/4} L_t^{5k/2}}^k \|D_x^{s_k} u\|_{L_x^5 L_t^{10}} + \|u\|_{L_x^{5k/4} L_t^{5k/2}} \|D_x^{s_k} u\|_{L_x^5 L_t^{10}} \|u^{k-1}\|_{L_x^{p_1} L_t^{q_1}}, \end{aligned} \quad (3.32)$$

where

$$\frac{1}{p_1} = \frac{1}{p_0} - \frac{1}{5} = 1 - \frac{4}{5k} - \frac{1}{5} = \frac{4(k-1)}{5k} \quad \text{and} \quad \frac{1}{q_1} = \frac{1}{q_0} - \frac{1}{10} = \frac{1}{2} - \frac{2}{5k} - \frac{1}{10} = \frac{4(k-1)}{10k}.$$

Therefore

$$\begin{aligned} \|\Phi(u)\|_{L_x^{5k/4} L_t^{5k/2}} &\leq \|U(t)u_0\|_{L_x^{5k/4} L_t^{5k/2}} + ca^k b \\ &\leq \delta + ca^k b. \end{aligned} \quad (3.33)$$

On the other hand, using the inhomogeneous smoothing effect (2.24), Lemma 2.6 with $(p_1, q_1, \alpha_1) = (5, 10, 0)$, $(p_2, q_2, \alpha_2) = (\infty, 2, 1)$, and (2.22) with $(p, q, \alpha) = (5, 10, 0)$, estimate (3.32) also implies

$$\begin{aligned} \|D_x^{s_k} \Phi(u)\|_{L_t^\infty L_x^2} + \|D_x^{s_k} \Phi(u)\|_{L_x^5 L_t^{10}} &\leq cK + c\|D_x^{s_k} U(t)u_0\|_{L_x^5 L_t^{10}} + c\|D_x^{s_k}(u^{k+1})\|_{L_x^1 L_t^2} \\ &\leq cK + ca^k b. \end{aligned} \quad (3.34)$$

First one chooses $b = 2cK$ and a such that $ca^k \leq 1/2$. Therefore, once for all

$$\|D_x^{s_k} \Phi(u)\|_{L_t^\infty L_x^2} + \|D_x^{s_k} \Phi(u)\|_{L_x^5 L_t^{10}} \leq b.$$

Finally, if one chooses $\delta = a/2$ and a so that $ca^{k-1}b \leq 1/2$ then we also have

$$\|\Phi(u)\|_{L_x^{5k/4} L_t^{5k/2}} \leq a.$$

Such calculations establish that $\Phi : X_{a,b}^k \rightarrow X_{a,b}^k$ is well defined. For the contraction one uses similar arguments. The contraction mapping principle then imply the existence of a unique fixed point for Φ , which is a solution of (1.10). The proof is completed with standard arguments.

To prove (1.13), one defines

$$f_\pm = u_0 + \mu \int_0^{\pm\infty} U(-t') \partial_x(u^{k+1})(t') dt'.$$

Then, as in (3.32)-(3.34), we have

$$\|D_x^{s_k}(u(t) - U(t)f_+)\|_{L_x^2} \leq \|D_x^{s_k} \int_t^{+\infty} U(t-t') \partial_x(u^{k+1})(t') dt'\|_{L_x^2} \lesssim \|u\|_{L_x^{5k/4} L_{[t,+\infty)}^{5k/2}}^k \|D_x^{s_k} u\|_{L_x^5 L_{[t,+\infty)}^{10}}.$$

Since $\|u\|_{L_x^{5k/4} L_t^{5k/2}} < \infty$ and $\|D_x^{s_k} u\|_{L_x^5 L_t^{10}} < \infty$, the above inequality implies (1.13). A similar calculation holds for f_- . \square

Theorem 1.3 follows from similar arguments, so it will be omitted. Next, we prove the blow-up result stated in Theorem 1.4.

Proof of Theorem 1.4. The proof is based on the arguments in [10, Theorem 1.2]. We first claim that if $\|u\|_{L_x^{5k/4} L_{[0,T^*]}^{5k/2}} < \infty$ then $\|D_x^{s_k} u\|_{L_x^5 L_{[0,T^*]}^{10}} < \infty$. Indeed, let $\varepsilon_0 > 0$ be an arbitrary small number. Since the norm $\|u\|_{L_x^{5k/4} L_{[0,T^*]}^{5k/2}}$ is finite we can split the interval $[0, T^*]$ in $0 = t_0 < t_1 < \dots < t_\ell = T^*$ such

that $\|u\|_{L_x^{5k/4} L_{I_n}^{5k/2}} < \varepsilon_0$, where $I_n = [t_n, t_{n+1}]$, $n = 0, 1, \dots, \ell - 1$. Since u is a fixed point of the integral equation (3.31), from (2.22) with $(p, q, \alpha) = (5, 10, 0)$, we have, for $n = 0, 1, \dots, \ell - 1$,

$$\begin{aligned} \|D_x^{s_k} u\|_{L_x^5 L_{I_n}^{10}} &\lesssim \|D_x^{s_k} u_0\|_{L_x^2} + \|D_x^{s_k} \int_0^t U(t-t') \partial_x(u^{k+1})(t') dt'\|_{L_x^5 L_{I_n}^{10}} \\ &\lesssim \|D_x^{s_k} u_0\|_{L_x^2} + \sum_{j=0}^n \|D_x^{s_k} \int_{t_j}^{t_{j+1}} U(t-t') \partial_x(u^{k+1})(t') dt'\|_{L_x^5 L_{I_n}^{10}} \\ &\lesssim \|D_x^{s_k} u_0\|_{L_x^2} + \sum_{j=0}^n \|D_x^{s_k} \int_0^t U(t-t') \partial_x(u^{k+1})(t') \chi_{I_j}(t') dt'\|_{L_x^5 L_t^{10}}, \end{aligned} \quad (3.35)$$

where χ_{I_j} denotes the characteristic function of the interval I_j . By using Lemma 2.6, with $(p_1, q_1, \alpha_1) = (5, 10, 0)$ and $(p_2, q_2, \alpha_2) = (\infty, 2, 1)$, we then deduce

$$\|D_x^{s_k} u\|_{L_x^5 L_{I_n}^{10}} \lesssim \|D_x^{s_k} u_0\|_{L_x^2} + \sum_{j=0}^n \|D_x^{s_k}(u^{k+1})\|_{L_x^1 L_{I_j}^2}. \quad (3.36)$$

Now, inequality (3.32) yields

$$\begin{aligned} \|D_x^{s_k} u\|_{L_x^5 L_{I_n}^{10}} &\lesssim \|D_x^{s_k} u_0\|_{L_x^2} + \sum_{j=0}^n \|D_x^{s_k} u\|_{L_x^5 L_{I_j}^{10}} \|u\|_{L_x^{5k/4} L_{I_j}^{5k/2}}^k \\ &\leq c \|D_x^{s_k} u_0\|_{L_x^2} + c \varepsilon_0^k \sum_{j=0}^n \|D_x^{s_k} u\|_{L_x^5 L_{I_j}^{10}}. \end{aligned} \quad (3.37)$$

Therefore, choosing $c \varepsilon_0^k < 1/2$, we conclude

$$\|D_x^{s_k} u\|_{L_x^5 L_{I_n}^{10}} \leq 2c \|D_x^{s_k} u_0\|_{L_x^2} + 2 \sum_{j=0}^{n-1} \|D_x^{s_k} u\|_{L_x^5 L_{I_j}^{10}}. \quad (3.38)$$

Inequality (3.38) together with an induction argument implies that $\|D_x^{s_k} u\|_{L_x^5 L_{I_n}^{10}} < \infty$ for $n = 0, 1, \dots, \ell - 1$. By summing over the ℓ intervals we conclude the claim.

Next we prove (1.15). Assume that the solution exists for $|t| < T'$ with $\|u\|_{L_x^{5k/4} L_{[0, T']}^{5k/2}} < \infty$, by the above claim we also have $\|D_x^{s_k} u\|_{L_x^5 L_{[0, T']}^{10}} < \infty$. Moreover, by inequality (3.32), we conclude for all $t \in [0, T']$

$$\|D_x^{s_k} u(t)\|_{L_x^2} \lesssim \|D_x^{s_k} u_0\|_{L_x^2} + \|u\|_{L_x^{5k/4} L_{[0, T']}^{5k/2}}^k \|D_x^{s_k} u\|_{L_x^5 L_{[0, T']}^{10}} < \infty.$$

As a consequence, $u(T') \in \dot{H}^{s_k}(\mathbb{R})$, which from Theorem 1.3 implies the existence of $\delta > 0$ such that the solution exists for $|t| \leq T' + \delta$. Thus, if $T^* < \infty$, (1.15) must be true.

Next, we turn to proof of (1.16). Let $\delta > 0$ be a small constant to be chosen later. Since $\|u\|_{L_x^{5k/4} L_{[0, T^*]}^{5k/2}} = \infty$, we can construct a family of intervals $I_n = [t_n, t_{n+1}]$ such that $t_n < t_{n+1}$, $t_n \nearrow T^*$, and

$$\|u\|_{L_x^{5k/4} L_{I_n}^{5k/2}} = \delta. \quad (3.39)$$

From analytic interpolation, we obtain

$$\|u\|_{L_x^{5k/4} L_{I_n}^{5k/2}} \lesssim \|D_x^{2/3k} u\|_{L_x^{3k/2} L_{I_n}^{3k/2}}^{3/5} \|D_x^{-1/k} u\|_{L_x^k L_{I_n}^\infty}^{2/5}. \quad (3.40)$$

On the other hand, since u is a fixed point of integral equation (3.31), for $t \in I_n$ we also have

$$u(t) = U(t - t_n)u(t_n) - \mu \int_{t_n}^t U(t - t') \partial_x(u^{k+1})(t') dt'. \quad (3.41)$$

Therefore

$$\|D_x^{-1/k} u\|_{L_x^k L_{I_n}^\infty} \lesssim \|D_x^{-1/k} U(t - t_n)u(t_n)\|_{L_x^k L_{I_n}^\infty} + \|D_x^{-1/k} \int_{t_n}^t U(t - t') \partial_x(u^{k+1})(t') dt'\|_{L_x^k L_{I_n}^\infty}.$$

To bound the linear part we use Corollary 2.4 and to bound the integral part, we use Lemma 2.6-(2.30) with $(p_2, \alpha_2) = (k, 1/2 - 1/k)$ and $(p_1, q_1, \alpha_1) = (\infty, 2, 1)$. Hence,

$$\begin{aligned} \|D_x^{-1/k} u\|_{L_x^k L_{I_n}^\infty} &\lesssim \|D_x^{s_k} U(-t_n)u(t_n)\|_{L_x^2} + \|D_x^{s_k}(u^{k+1})\|_{L_x^1 L_{I_n}^2} \\ &\leq cK + c\delta^k \|D_x^{s_k} u\|_{L_x^5 L_{I_n}^{10}}, \end{aligned}$$

where in the last inequality we have used $\sup_{t \in [0, T^*)} \|D_x^{s_k} u(t)\|_{L_x^2} = K$ and (3.39).

It remains to bound the norm $\|D_x^{s_k} u\|_{L_x^5 L_{I_n}^{10}}$. Indeed using again the integral equation (3.41), Lemma 2.6 with $(p_1, q_1, \alpha_1) = (5, 10, 0)$, $(p_2, q_2, \alpha_2) = (\infty, 2, 1)$ and estimate (3.32) we obtain

$$\begin{aligned} \|D_x^{s_k} u\|_{L_x^5 L_{I_n}^{10}} &\lesssim \|D_x^{s_k} U(-t_n)u(t_n)\|_{L_x^2} + \|D_x^{s_k} u\|_{L_x^5 L_{I_n}^{10}} \|u\|_{L_x^{5k/4} L_{I_n}^{5k/2}}^k \\ &\leq cK + c\delta^k \|D_x^{s_k} u\|_{L_x^5 L_{I_n}^{10}}. \end{aligned} \quad (3.42)$$

Hence, choosing $c\delta^k < 1/2$ we conclude $\|D_x^{s_k} u\|_{L_x^5 L_{I_n}^{10}} \leq 2cK$, which also implies $\|D_x^{-1/k} u\|_{L_x^k L_{I_n}^\infty} \leq 2cK$. The last two inequalities combining with (3.40) yield

$$\|D_x^{2/3k} u\|_{L_x^{3k/2} L_{I_n}^{3k/2}} \geq \frac{\delta^{5/3}}{(2cK)^{2/3}}, \text{ for all } n \in \mathbb{N}$$

and, therefore (1.16) holds. \square

4. THE CONSTRUCTION OF THE WAVE OPERATOR

In this section, we intend to show Theorem 1.6. Following the ideas introduced by Côte [2], we must look for a fixed point for the operator

$$\Phi : w(t) \longrightarrow -\mu \partial_x \int_t^\infty U(t - t')(w(t') + U(t')v)^{k+1} dt',$$

defined in the time interval $[T_0, \infty)$, where $T_0 > 0$ is an arbitrarily large number that will be chosen later.

The next proposition says that this fixed point provides a function u satisfying the integral equation (1.1) (we refer to Farah [3] for a proof).

Proposition 4.1. *Let w be a fixed point of the operator Φ and define*

$$u(t) = U(t)v + w(t). \quad (4.43)$$

Then u is a solution of (1.1) in the time interval $[T_0, \infty)$.

Proof of Theorem 1.6. Again we use the contraction mapping principle. Given $T > 0$, define the metric spaces

$$X_T = \{w \in C(\mathbb{R}; \dot{H}^{s_k}(\mathbb{R})); \|w\|_{X_T} < \infty\}$$

and

$$X_T^a = \{w \in X_T; \|w\|_{X_T} \leq a\},$$

where

$$\|w\|_{X_T} = \|w\|_{L_T^\infty \dot{H}^{s_k}} + \|D_x^{s_k} w\|_{L_x^5 L_T^{10}} + \|w\|_{L_x^{5k/4} L_T^{5k/2}}.$$

Applying Corollary 2.7 we obtain

$$\|\Phi(w)\|_{L_x^{5k/4} L_T^{5k/2}} \lesssim \|D_x^{s_k}(U(t)v + w(t))^{k+1}\|_{L_x^1 L_T^2}.$$

Using the same arguments as the ones used in (3.32), we conclude

$$\|D_x^{s_k}(U(t)v + w(t))^{k+1}\|_{L_x^1 L_T^2} \lesssim \|U(t)v + w(t)\|_{L_x^{5k/4} L_T^{5k/2}}^k \|D_x^{s_k}(U(t)v + w(t))\|_{L_x^5 L_T^{10}}.$$

The other norms can be estimated in the same manner, which implies

$$\begin{aligned} \|\Phi(w)\|_{X_T} &\lesssim \|U(t)v + w(t)\|_{L_x^{5k/4} L_T^{5k/2}}^k \|D_x^{s_k}(U(t)v + w(t))\|_{L_x^5 L_T^{10}} \\ &\lesssim \|U(t)v + w(t)\|_{L_x^{5k/4} L_T^{5k/2}}^k \|D_x^{s_k}(U(t)v + w(t))\|_{L_x^5 L_T^{10}} \\ &\lesssim \|U(t)v\|_{L_x^{5k/4} L_T^{5k/2}}^k \|D_x^{s_k} v\|_{L_x^2} + \|D_x^{s_k} v\|_{L_x^2} \|w\|_{X_T}^k + \|w\|_{X_T}^{k+1}, \end{aligned} \quad (4.44)$$

where in the last inequality we have used Lemma 2.1 and Lemma 2.5.

Since $\|U(t)v\|_{L_x^{5k/4} L_T^{5k/2}} \rightarrow 0$ as $T \rightarrow \infty$ we can find a $T_0 > 0$ large enough and $a > 0$ small enough such that $\Phi : X_{T_0}^a \rightarrow X_{T_0}^a$ is well defined and is a contraction. Therefore, Φ has a unique fixed point, which we denote by w . Moreover, $a > 0$ can be chosen such that

$$\|w\|_{X_T} \lesssim \|U(t)v\|_{L_x^{5k/4} L_T^{5k/2}}^k \|D_x^{s_k} v\|_{L_x^2}, \quad (4.45)$$

which implies $\|w\|_{X_T} \rightarrow 0$ as $T \rightarrow \infty$.

Next we show the limit (1.17). By the Proposition, $u(t) = U(t)v + w(t)$ satisfies the integral equation in the right hand side of (3.31) in the time interval $[T_0, \infty)$. Therefore

$$\begin{aligned} \|u(t) - U(t)v\|_{L_T^\infty \dot{H}^{s_k}} &= \|w\|_{L_T^\infty \dot{H}^{s_k}} = \|\Phi(w)\|_{L_T^\infty \dot{H}^{s_k}} \\ &\lesssim \|U(t)v\|_{L_x^{5k/4} L_T^{5k/2}}^k \|D_x^{s_k} v\|_{L_x^2} + \|D_x^{s_k} v\|_{L_x^2} \|w\|_{X_T}^k + \|w\|_{X_T}^{k+1}. \end{aligned}$$

The last inequality together with (4.45) implies (1.17), finishing the proof of the theorem. \square

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